C. Manolescu - 30/5/08 - Symplectic instanton homology (w/ Woodward)

<u>Instanton homology</u> (Floer): → target for relative Donaldson invts

Y^3, $H_1(Y)=0$ ⇝ CF_* gen^d by <u>irred</u>. representations

$$\{\rho:\pi_1(Y)\to SU(2)\}/_{PSU(2)} \equiv \{A\in\mathcal{A}(Y) / F_A=0\}/\mathcal{G}$$

conns. on $Y\times SU(2)\to Y$

($\rho:\pi_1(Y)\to SU(2)$ is irreducible ⟺ conj. orbit $\mathcal{O}_\rho = PSU(2)$

trivial $\rho=1$ ⇒ $PSU(2)$-conj. orbit: $\mathcal{O}_\rho = pt$)

∂: CF_* ↺ $\quad \partial[A] = \sum_B n_{AB}[B]$, n_{AB} Counts sol^{ns} to ASD eq^{ns} on $Y\times\mathbb{R}$

⇝ $HF_*(Y)$ $\mathbb{Z}/8$ - graded homology

<u>Versions of HF_*</u>: (Donaldson's book)

		irred	red	
Floer	HF	$\mathbb{Z}$	0	
Austin-Braam	$\underline{\underline{HF}}$	$\mathbb{Q}$	$\mathbb{Q}[t]$ deg $t=4$	← $H^*_{PSU(2)}(\mathcal{O}_\rho,\mathbb{Q})$
"framed instanton h_*"	$\widetilde{HF}$	$\mathbb{Q}\oplus\mathbb{Q}$	$\mathbb{Q}$	← $H^*(\mathcal{O}_\rho,\mathbb{Q})$

<u>Questions</u>: 1) how about <u>HF for other 3-manifolds</u>?

<u>Issue</u>: Nontrivial reducibles: $\rho:\pi_1(Y)\to S^1\to SU(2)$

conj. orbit $\mathcal{O}_\rho \simeq S^2$

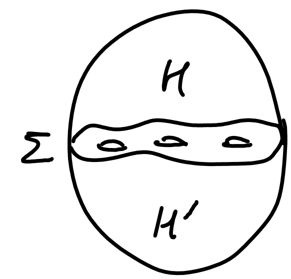
Could instead use $\begin{matrix}E\\ \downarrow\\ Y\end{matrix}$ nontrivial $U(2)$-bundle with c_1 odd

⇒ no reducibles ✓

Can one define $\underline{\underline{HF}}$, $\widetilde{HF}$ for any Y?

2) Atiyah-Floer conjecture: $Y = H \cup_{\Sigma} H'$

handlebodies



$$\begin{array}{l} \mathcal{M}(H) \hookrightarrow \mathcal{M}(\Sigma) \\ \mathcal{M}(H') \nearrow \end{array}$$

"Lagrangian" "symplectic" $\rightsquigarrow$ "$HF(Y) = HF_{Lag}(\mathcal{M}(H), \mathcal{M}(H'))$"

Lag. Floer homology in $\mathcal{M}(\Sigma)$.

Problem: $\mathcal{M}(H)$, $\mathcal{M}(H')$, $\mathcal{M}(\Sigma)$ are all singular!

Works for nontrivial bundles

($\rightsquigarrow$ A-F conj. proved for mapping tori - Dostoglou-Salamon)

See also Wehrheim-Woodward ...

Today: trivial $SU(2)$-bundles

Given any $Y^3 = H \cup_{\Sigma} H'$, define $\widetilde{HSI}_*(\Sigma, H, H')$ $\mathbb{Z}/8$-graded, $\mathbb{Z}$-coefficients

(defined using Lagrangian HF)

Conj: This is a 3-manifold invariant (indep.t of Heegaard splitting)

Conj: (Atiyah-Floer): $H_1(Y) = 0 \Rightarrow \widetilde{HSI} = \widetilde{HF}$

Work in progress:

- G simply connected simple Lie group (ADE type) $\rightsquigarrow \widetilde{HSI}_G$ ($G \neq SU(2)$: inv.ce ???)
- equivariant HSI (cf. Viterbo - Bourgeois - Oancea)

Goal: 3+1 TQFT

Recall $\quad \mathcal{M}(\Sigma) = \left\{ A \in \mathcal{A}\left(\begin{smallmatrix}\Sigma \times SU(2) \\ \downarrow \\ \Sigma\end{smallmatrix}\right) \Big/ F_A = 0 \right\} \Big/ \text{gauge}$

$$= \{ \rho : \pi_1 \Sigma \to SU(2) \} / PSU(2)$$

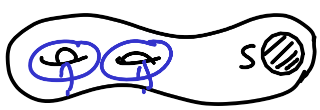
$$= \{ (A_1, B_1, \dots, A_g, B_g) \in SU(2)^{2g} / \prod [A_i, B_i] = 1 \} / PSU_2$$

$(6g-6)$-dim^l manifold,
singular at reducibles $\{ \rho : \pi_1 \Sigma \to S^1 \} \cong T^{2g} / \mathbb{Z}_2$

The smooth locus of $\mathcal{M}(\Sigma)$ carries a symplectic form

$$\omega(a,b) = \int_\Sigma Tr(a \wedge b)$$

- Framed connections: extended moduli space (Jeffrey, Huebschman)

Σ 

$\rightsquigarrow \Sigma' = \Sigma \setminus D^2, \quad S = \partial D^2 = \partial \Sigma'.$

$$\mathcal{M}^{\mathfrak{g}}(\Sigma') := \left\{ A \in \mathcal{A}(\Sigma') \Big/ \begin{matrix} F_A = 0, & A = \theta\, ds \text{ near } S \\ & \theta \in \mathfrak{g} = \text{Lie } G \end{matrix} \right\} \Big/ \left\{ \begin{matrix} f : \Sigma' \to SU(2) \\ f = I \text{ near } S \end{matrix} \right\}$$

$$= \{ (A_1, B_1, \dots, A_g, B_g) \in SU(2)^{2g}, \ \theta \in \underline{su}(2) / \prod [A_i, B_i] = \exp \theta \}$$

$\mathcal{M}^{\mathfrak{g}}(\Sigma') \circlearrowleft SU(2)$ residual action (gauge group at S)

$$\Phi : \mathcal{M}^{\mathfrak{g}}(\Sigma') \to su(2)$$
$$(A_i, B_i, \theta) \mapsto \theta$$

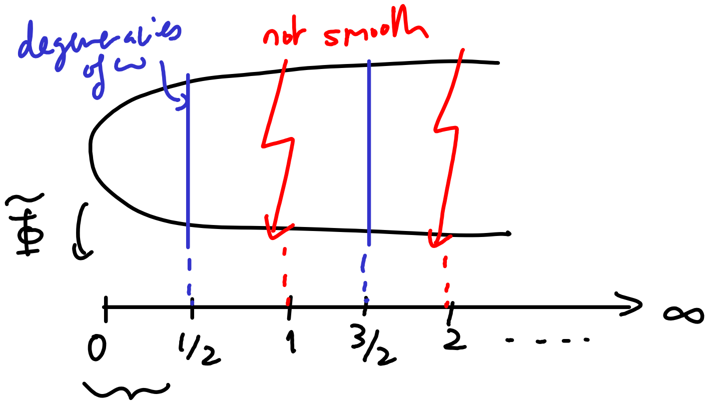
$Q : \underline{su}(2) \longmapsto [0, \infty) \qquad Q(\theta) = t$ if $\theta \underset{conj.}{\sim} t \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$\tilde{\Phi} := Q \circ \Phi : \mathcal{M}^{\mathfrak{g}}(\Sigma') \to [0, \infty)$

Note: • $\Phi^{-1}(0)/G = \mathcal{M}(\Sigma)$

• $\mathcal{M}^{\mathfrak{g}}(\Sigma')$ is $\begin{cases} \text{• smooth outside } \tilde{\Phi}^{-1}(\mathbb{Z}_{>0}) \\ \text{• symplectic outside } \tilde{\Phi}^{-1}(\frac{1}{2}\mathbb{Z}_{>0}) \end{cases}$

$\omega \in \Omega^2(\mathcal{M}^{\mathfrak{g}}(\Sigma'))$ closed, but degenerate at $\tilde{\Phi} \in \frac{1}{2}\mathbb{Z}$



• The $SU(2)$-action on $\mathcal{M}^{\mathfrak{g}}(\Sigma') \setminus (\dots)$ is Hamiltonian and its moment map is Φ

Let $\mathcal{U} = \tilde{\Phi}^{-1}([0,\frac{1}{2}))$: then $\mathcal{M}(\Sigma) \cong \phi^{-1}(0)/G \cong \mathcal{U}/\!/G$.

Now, if $Y = H \cup_{\Sigma} H'$ handlebodies

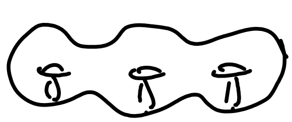
$$L(H) := \{\pi_1(H) \to SU(2)\} = SU(2)^g \subset \Phi^{-1}(0) \subset \mathcal{M}^{\mathfrak{g}}(\Sigma')$$

$\pi_1(H)$ = free gp $\mathbb{F}_g$

$$= \{A_i = Id,\ \theta = 0\}$$

$L(H')$ similarly

Ex: S^3 :



$L(H) = \{A_i = Id\}$
$L(H') = \{B_i = id\}$

Ideally: $\widetilde{HSI} := HF_*(L(H), L(H') \text{ in } \mathcal{U})$

NB: $\mathcal{U}$ is monotone

Problem: $\mathcal{U}$ is not compact, not convex at ∞.

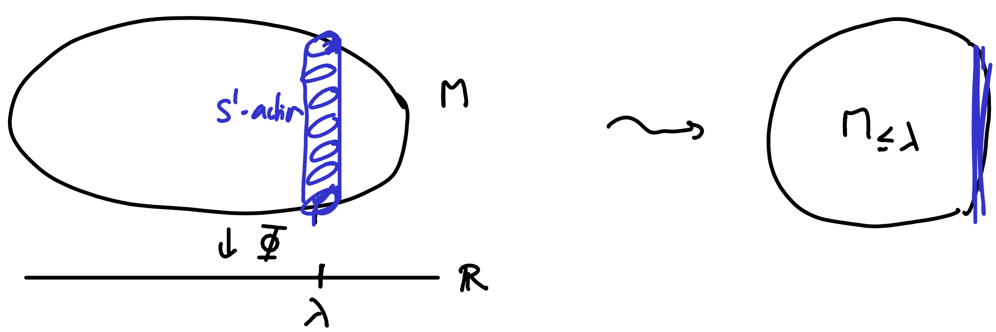
$\Rightarrow$ compactify $\mathcal{U}$ by cutting.

- Abelian symplectic cutting (Lerman)

$(M,\omega) \circlearrowleft$ Ham. S^1-action, $\phi: M \to \mathbb{R}$ moment map
$\lambda \in \mathbb{R}$

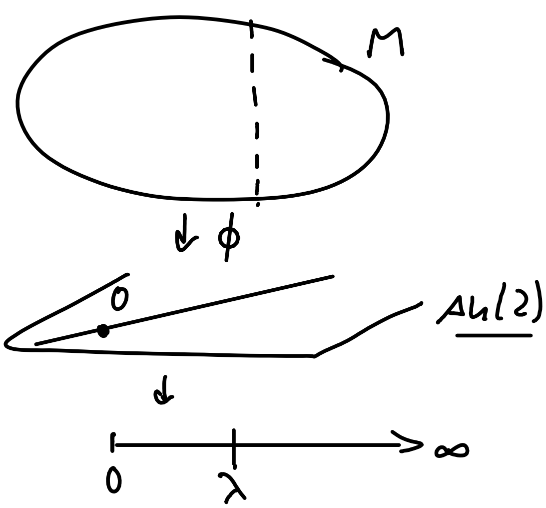
Consider $M \times \mathbb{C}$, diagonal S^1-action, $\psi(m,z) = \phi(m) + \frac{|z|^2}{2}$

$\rightsquigarrow M_{\leq\lambda} := \psi^{-1}(\lambda)/S^1 \simeq \phi^{-1}(-\infty,\lambda) \cup \left(\phi^{-1}(\lambda)/S^1\right)$



- Nonabelian cutting (Woodward)

$(M,\omega) \circlearrowleft SU(2)$ Ham. action, $\lambda \in (0,\infty)$



$\rightsquigarrow S^1$-action on $M \setminus \phi^{-1}(0)$:

$u = e^{i\theta}$ acts by $m \mapsto \exp\left(\theta \frac{\phi(m)}{\tilde{\phi}(m)}\right) m$

($\tilde{\phi} = |\phi|$ as before)

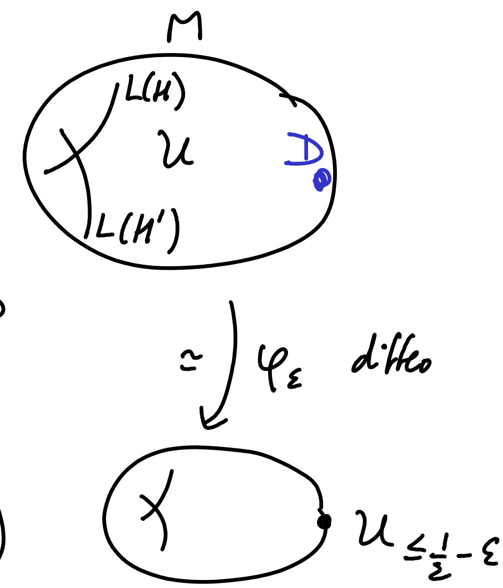
moment map is $\tilde{\Phi}$.

$\Rightarrow$ $M_{\leq\lambda} := \phi^{-1}(0) \cup \left(\tilde{\phi}^{-1}(0,\infty)\right)_{\leq\lambda}$

S^1-cutting

Apply to our setting $\Rightarrow \mathcal{U}_{\leq\lambda}$ compact, monotone iff $\lambda = \frac{1}{2}$.

Let $M = \mathcal{U}_{\leq \frac{1}{2}} = \mathcal{U} \cup D$
$\uparrow$
$\tilde{\Phi}^{-1}([0,\tfrac{1}{2}))$



$\rightsquigarrow (M, \omega_0)$, monotone but
ω_0 degenerate along D

But $M \underset{\varphi_\varepsilon}{\overset{\text{diffeo}}{\xrightarrow{\sim}}} \mathcal{U}_{\leq \frac{1}{2} - \varepsilon}$

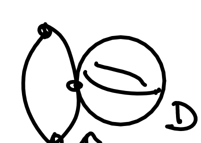
$\Rightarrow \omega_\varepsilon = \varphi_\varepsilon^* \, \omega_{\leq \frac{1}{2} - \varepsilon}$ symplectic (nondeg.)
but not monotone

$\rightsquigarrow$ Define $HF(L, L')$ using $\tilde{J}$ a.c.s. that
$$\begin{cases} \text{tames } \omega_\varepsilon \\ \text{tames } \omega_0 \text{ outside } D \\ \omega_0(v, \tilde{J} v) \geq 0 \text{ on } D \end{cases}$$

Thm: ∂ count J_t-holom. disks rel. L_0, L_1 (outside D)
where J_t = ω_ε-tame generic perturb^n of $\tilde{J}$
$\Rightarrow \partial$ is finite and $\partial^2 = 0$.

Pf: • energy bound $\Rightarrow$ J_t-hol. objects have index ≥ 0
(index $= \int \omega_0$ by monotonicity)
(true for $\tilde{J} \Rightarrow$ true for J_t)

• bubbles of index 0 are $S^2 \subset D$



codim. 4 $\Rightarrow \partial^2 = 0$

Rmk: • for $G \neq SU(2)$, codim $D = 2 \cdot \operatorname{rank}(G)$
• invariance? – need more quilt theory

Open q^n: relate $\widetilde{HSI}$ to Ozsvath-Szabo's $\widehat{HF}$? (same rk for S^3, $S^2 \times S^1$, $L(p,1)$)